The “Four Times the Impedance” Rule for Broadband RF Transformer Windings: Where Does it Originate?

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Abstract: *Broadband transformers are widely used in RF applications for impedance matching, isolation, and coupling where a large range of operating frequencies must be accommodated. Design methods for these types of transformers are well-known and widely published. A popular rule-of-thumb for the design of these transformers is that the inductive reactance of the windings must be at least four times larger than the impedance to which the windings are connected. This paper explains the operation of transformer circuits and shows that a transformer designed with the “Four Times the Impedance” rule functions to within three percent of an “ideal” transformer.*

Introduction

The use of broadband ferrite transformers in RF circuitry for impedance transformation, isolation, and inter-stage coupling is widespread, and any radio amateur who enjoys “homebrewing” projects has probably fabricated at least one of these devices at one time or another. Exceedingly simple to construct with nothing more than a ferrite toroid and some copper wire, these types of transformers can be found in baluns, mixers, oscillators, amplifiers, or any other circuit where RF energy needs to be coupled from one place to another. In addition to being excellent coupling devices, broadband transformers are well-known for their ability to transform impedances over a wide frequency range. This property is essential in many aspects of RF system design as it is well known that maximum transfer of power occurs when the source impedance of the driving circuit is matched to the impedance of the load being driven. Although impedances may be matched with “resonant-mode” lumped-parameter circuits like L-sections and Pi-sections, these circuit topologies only provide reasonable impedance matching over a narrow frequency range. As transformers do not rely on the properties of resonant circuits for their impedance matching function, they can be used as effective matching devices over very wide ranges of frequency.

The figure below illustrates a typical broadband RF transformer:

Figure 1 - A Broadband RF Transformer

A toroid comprised of powdered iron or some other special mixture of ferrite material is wrapped with a pair (or several pairs) of copper conductors in such a way so that the magnetic fields associated with each of the windings are very tightly coupled. Windings are often twisted together in a “bifilar” manner to further increase their coupling to each other. The material comprising the toroid generally has a very high magnetic permeability, which has the effect of concentrating the magnetic field from the windings inside the toroid itself. This has two beneficial effects. First, coupling between the windings is improved because the toroid “channels” the field inside the windings. Second, containing the magnetic field inside the toroid makes the transformer relatively immune from coupling to other sources of magnetic fields. This allows toroids to be placed in very close proximity to each other on a printed circuit board without fear of interaction between them.

A broadband transformer used in an impedance matching application is usually presented in the following manner: a load impedance Z_L is connected to the secondary winding of the transformer. When viewed through the primary winding, the magnitude of the impedance is altered by the square of the primary-to-secondary turns ratio. The design equation which describes this is well-known to all radio amateurs:

$$Z_{in} = \left(\frac{N_P}{N_S} \right)^2 Z_L \qquad (1.1)$$

Where N_P and N_S are the number of turns in the primary and secondary windings, respectively. The broadband nature of the impedance transformation is clear from the equation – nowhere does frequency enter the picture! The load impedance transformation is solely a function of the primary-to-secondary turns ratio. It is important to remember that this equation for input impedance applies to an “ideal” transformer – one that has zero leakage inductance and zero ohmic loss.

When designing a broadband RF transformer, one parameter to consider is the number of turns that the primary and secondary windings must contain in order for the behavior of the transformer to be nearly “ideal”. There are a number of sources available to the radio amateur to assist in this aspect of the design. The ARRL Handbook for Radio Communications 2012 edition states *“One of the most common ferromagnetic transformers used in amateur circuits is the conventional broadband transformer. Broadband transformers with losses of less than 1 dB are employed in circuits that must have a uniform response over a substantial frequency range, such as a 2- to 30-MHz broadband amplifier. In applications of this sort, the reactance of the windings should be at least four times the impedance that the winding is designed to look into at the lowest design frequency”* (American Radio Relay League, 2012, p. 5.28). In the article “Construction and Use of Broadband Transformers”, the author writes *“Most industrial designers follow a rule that dictates the XL (inductive reactance) of the smallest winding must be X4* [four times] *or greater the load impedance of the winding.”* (Unknown, 2012). Finally, a Motorola application note entitled “Broadband Transformers and Power Combining Techniques for RF” (Granberg, 1993) presents an equation for the minimum value of inductance on the low impedance side of a broadband transformer that is consistent with the “four times the impedance” rule. Other sources not specifically mentioned here also exist that utilize this same design rule-of-thumb, however none of the articles reviewed by this author devote any attention to justifying this design guideline or cite a reference from where it originated. The purpose of this article is to analyze the behavior of an RF transformer with finite winding reactance, and place the “four times the impedance” rule on a firm and quantitative footing.

Circuit Analysis

We begin our analysis by modeling a broadband RF transformer as a pair of coupled inductors. Consider the circuit in the figure below:

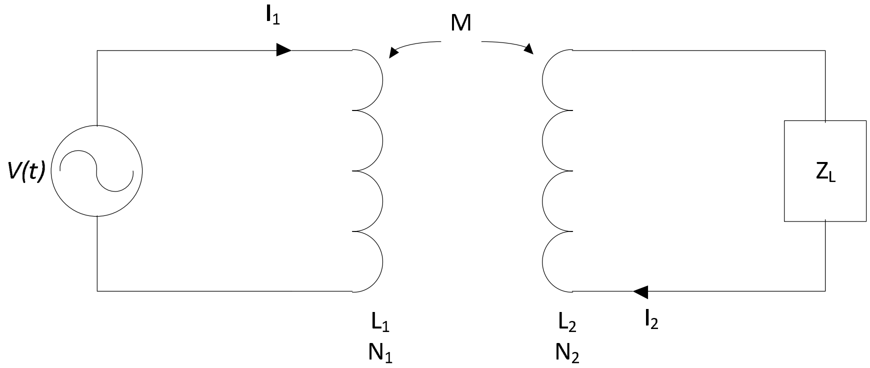

Figure 2 - Generalized Coupled Circuit

We seek the input impedance seen by the generator driving inductor L_1 when a load impedance Z_L is present in the secondary circuit. The loop equations for currents I_1 and I_2 are:

$$\begin{aligned} V(t) &= L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \\ 0 &= L_2 \frac{di_2}{dt} + M\frac{di_1}{dt} + Z_L i_2 \end{aligned} \tag{2.1}$$

Since we are considering only sinusoidal signals in our analysis, we may rewrite these loop equations as:

$$\begin{aligned} V &= j\omega L_1 i_1 + j\omega M i_2 \\ 0 &= j\omega L_2 i_2 + j\omega M i_1 + Z_L i_2 \end{aligned} \tag{2.2}$$

Solving the second equation for I_2, we find:

$$i_2 = -\frac{j\omega M i_1}{(Z_L + j\omega L_2)} \tag{2.3}$$

Substituting this back into the first equation we solve for the input impedance seen by the generator:

$$Z_{in} = \frac{V}{i_1} = j\omega L_1 + \frac{\omega^2 M^2}{\left(Z_L + j\omega L_2\right)} \tag{2.4}$$

Manipulating this slightly to clear the complex number out of the denominator, we have:

$$Z_{in} = \frac{\omega^2 M^2 Z_L}{\left(Z_L{}^2 + \omega^2 L_2{}^2\right)} + j\left[\frac{\omega L_1\left(Z_L{}^2 + \omega^2 L_2{}^2\right) - \omega^3 M^2 L_2}{\left(Z_L{}^2 + \omega^2 L_2{}^2\right)}\right] \tag{2.5}$$

Notice that the input impedance has a real part which is proportional to M^2, and if we allow the mutual inductance to vanish, Eq. (2.5) reduces to the inductive reactance of L_1.

In order to make better use of the input impedance expression we have just derived, it would be useful if we can find some way of expressing the mutual inductance in terms of the individual inductances present in the circuit. It is possible to do this in a relatively simple manner by considering Faraday’s Law and how it governs the behavior of two inductances that are coupled by a changing magnetic field. Recall that Faraday’s Law states that the electromotive force (or emf) induced across a coil containing *N* turns is equal to the time rate of change of magnetic flux passing through it. Putting this into a mathematical expression yields:

$$v = -N\frac{d\phi}{dt} \tag{2.6}$$

Where the negative sign is a consequence of *Lenz’s Law*, which states that the induced emf must be oriented to oppose the change in flux (this is a consequence of the law of conservation of energy). We can also write Faraday’s law in terms of the inductance of the coil and the current flowing in it:

$$v = -L\frac{di}{dt} \tag{2.7}$$

Where again, the negative sign indicates that the voltage appearing across an inductor whose current is changing is oriented so it tends to oppose the change in current. Equating these two expressions:

$$\begin{aligned} N\frac{d\phi}{dt} &= L\frac{di}{dt} \\ N\,d\phi &= L\,di \\ L &= N\frac{d\phi}{di} \end{aligned} \tag{2.8}$$

The last expression in the above equation is a useful one, as it expresses the inductance of a coil in terms of the derivative of magnetic flux with respect to current.

Let’s now consider two coils L_1 and L_2 which have N_1 and N_2 turns, respectively, as in the figure below:

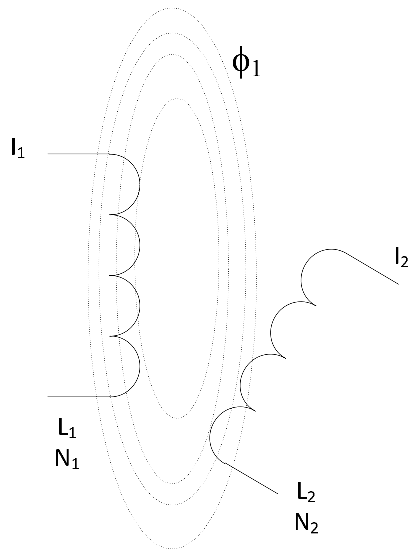

Figure 3 - Two Coils in Close Proximity

The current I_1 flowing in coil L_1 gives rise to a magnetic flux ϕ_1. This flux and the inductance of coil L_1 are related by Eq. (2.8) in the following way:

$$L_1 = N_1 \frac{d\phi_1}{di_1} \tag{2.9}$$

Because coil L_2 is in close proximity to L_1, a portion of the flux arising from current I_1 is linked to L_2. If flux ϕ_1 varies with time, an emf will be induced across coil L_2. This coupling can be accounted for by defining the mutual inductance between coils L_1 and L_2 as:

$$M_{21} = N_2 \frac{d\phi_{21}}{di_1} \tag{2.10}$$

where ϕ_{21} is defined as the portion of the flux ϕ_1 that links coil L_2. We can also write:

$$\phi_{21} = k\phi_1 \tag{2.11}$$

where the constant *k* is known as a *coefficient of coupling*, and can take on values $-1 \le k \le 1$. Negative values are possible because mutual inductance can be defined as positive or negative, depending upon the reference directions chosen for the induced emf’s on both of the coils. Using Eq. (2.9) and (2.11) we can rewrite Eq. (2.10) as:

$$\begin{aligned} M_{21} &= N_2 k \frac{d\phi_1}{di_1} \\ M_{21} &= \frac{N_2 k L_1}{N_1} \end{aligned} \tag{2.12}$$

We see that we have been able to express the mutual inductance in terms of coil L_1 alone, along with the turns ratio of the two coils.

We can proceed in an identical manner starting with coil L_2, and arrive at a result identical to Eq. (2.12) with the subscript indices reversed:

$$M_{12} = \frac{N_1 k L_2}{N_2} \tag{2.13}$$

We have made a reasonable assumption that the coefficient of coupling is the same with respect to either coil (in isotropic and homogeneous media this will be true). Along with this assumption we can also conclude that the mutual inductance between the coils is the same with respect to either one, so that $M_{12} = M_{21}$. Equating Eq. (2.12) and (2.13) gives us:

$$\begin{aligned} \frac{N_2 k L_1}{N_1} &= \frac{N_1 k L_2}{N_2} \\ N_2^{\,2} L_1 &= N_1^{\,2} L_2 \\ \frac{L_1}{L_2} &= \left(\frac{N_1}{N_2}\right)^2 \end{aligned} \tag{2.14}$$

This is an expected result, as we know that the inductance of a coil is proportional to the square of the number of turns. If we look at the square of the mutual inductance using either Eq. (2.12) or (2.13), we find:

$$\begin{aligned} M^2 &= \left(\frac{N_2}{N_1}\right)^2 k^2 L_1^{\,2} \\ M^2 &= \left(\frac{L_2}{L_1}\right) k^2 L_1^{\,2} \\ M^2 &= k^2 L_1 L_2 \\ M &= k\sqrt{L_1 L_2} \end{aligned} \tag{2.15}$$

This is the expression we seek – the mutual inductance between the two coils as a function of the individual inductances.

For the case of a transformer, we desire very tight coupling between the two coils, so that all of the flux arising from one coil is linked to the other. In this case, the coefficient of coupling is equal to unity, and we can write:

$$
\begin{aligned}
M &= \sqrt{L_1 L_2} \\
M &= \left(\frac{N_2}{N_1}\right) L_1 = \left(\frac{N_1}{N_2}\right) L_2
\end{aligned}
\tag{2.16}
$$

We can now use these expressions for mutual inductance in Eq. (2.5) for the input impedance. Proceeding in this fashion, we get:

$$
Z_{in} = \frac{\omega^2 \left(\frac{N_1}{N_2}\right)^2 L_2{}^2 Z_L}{(Z_L{}^2 + \omega^2 L_2{}^2)} + j\left[\frac{\omega L_1 (Z_L{}^2 + \omega^2 L_2{}^2) - \omega^2 \left(\frac{N_1}{N_2}\right)^2 L_2{}^2 \omega L_2}{(Z_L{}^2 + \omega^2 L_2{}^2)}\right]
\tag{2.17}
$$

This complicated expression for the input impedance can be simplified significantly if we consider the case where the inductive reactance of coil L_2 is much larger than the magnitude of the load impedance Z_L, i.e., $\omega^2 L_2{}^2 \gg Z_L{}^2$. Under these conditions, we find that:

$$
\begin{aligned}
Z_{in} &= \left(\frac{N_1}{N_2}\right)^2 Z_L + j\left[\omega L_1 - \left(\frac{N_1}{N_2}\right)^2 \omega L_2\right] \\
Z_{in} &= \left(\frac{N_1}{N_2}\right)^2 Z_L + j\left[\omega L_1 - \omega L_1\right] \\
Z_{in} &= \left(\frac{N_1}{N_2}\right)^2 Z_L
\end{aligned}
\tag{2.18}
$$

We have successfully found the relationship that applies to any "ideal" transformer – a load impedance is transformed in magnitude by the square of the primary-to-secondary turns ratio! We have also found the condition that must hold in order for this equation to be valid – the inductive reactance of the secondary winding must be much larger than the magnitude of the load impedance.

It is natural to ask at this point, "How large is 'much larger'?" It is this question that we seek to answer, so we return to our expression of input impedance and examine how it behaves as we vary the inductive reactance in the windings. As is usual in analyses of this type, we will find it useful to express all the relevant quantities we wish to study in terms of ratios of inductive reactance to load impedance. We consider Eq. (2.17) and rearrange it in the following way:

$$Z_{in} = \left(\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right)\left(\frac{\left(\frac{N_1}{N_2}\right)^2}{\left[1+\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right]}\right) Z_L + j\left[\omega L_1 - \left(\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right)\left(\frac{\left(\frac{N_1}{N_2}\right)^2}{\left[1+\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right]}\right)\omega L_2\right]$$
$$Z_{in} = \left(\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right)\left(\frac{\left(\frac{N_1}{N_2}\right)^2}{\left[1+\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right]}\right) Z_L + j\left[\omega L_1 - \left(\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right)\left(\frac{1}{\left[1+\frac{\omega^2 L_2{}^2}{Z_L{}^2}\right]}\right)\omega L_1\right] \tag{2.19}$$

We define a dimensionless ratio $\varepsilon_2{}^2$:

$$\varepsilon_2{}^2 = \frac{\omega^2 L_2{}^2}{Z_L{}^2} \tag{2.20}$$

We can rewrite Eq. (2.19) as:

$$Z_{in} = \left(\frac{\varepsilon_2{}^2}{1+\varepsilon_2{}^2}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L + j\frac{\omega L_1}{(1+\varepsilon_2{}^2)} \tag{2.21}$$

We define another dimensionless ratio ε_1:

$$\varepsilon_1 = \frac{\omega L_1}{Z_L} \tag{2.22}$$

Plugging this into Eq. (2.21) results in:

$$Z_{in} = \left(\frac{\varepsilon_2{}^2}{1+\varepsilon_2{}^2}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L + j\frac{\varepsilon_1}{(1+\varepsilon_2{}^2)} Z_L \tag{2.23}$$

Applying a little bit of algebra to this, we find:

$$
\begin{aligned}
Z_{in} &= \left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L + j\frac{\varepsilon_1\varepsilon_2^{\,2}}{\varepsilon_2^{\,2}(1+\varepsilon_2^{\,2})} Z_L \\
Z_{in} &= \left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L + j\left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{\varepsilon_1}{\varepsilon_2^{\,2}}\right) Z_L \\
Z_{in} &= \left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L + j\frac{1}{\varepsilon_2}\left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 Z_L \\
Z_{in} &= \left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 \left\{1+\frac{j}{\varepsilon_2}\right\} Z_L
\end{aligned}
\tag{2.24}
$$

This last expression exposes to us the behavior of the input impedance of the transformer as a function of the ratio of the inductive reactance of the secondary winding to the load impedance. We see that as the ratio ε_2 becomes infinitely large, the input impedance reduces to Eq. (2.18), the "ideal" transformer case. However, Eq. (2.24) allows us to explore the departure from ideal transformer behavior and to justify in a quantitative way the "rule of thumb" regarding the inductive reactance of the windings.

If we normalize the input impedance to the "ideal" value, we have:

$$
\begin{aligned}
\frac{Z_{in}}{Z_{ideal}} &= \frac{\left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left(\frac{N_1}{N_2}\right)^2 \left\{1+\frac{j}{\varepsilon_2}\right\} Z_L}{\left(\frac{N_1}{N_2}\right)^2 Z_L} \\
\frac{Z_{in}}{Z_{ideal}} &= \left(\frac{\varepsilon_2^{\,2}}{1+\varepsilon_2^{\,2}}\right)\left\{1+\frac{j}{\varepsilon_2}\right\}
\end{aligned}
\tag{2.25}
$$

We will use this last expression to investigate the non-ideal behavior of the transformer. We expect that this normalized expression approaches unity with zero phase angle as ε_2 becomes infinitely large.

The graph below illustrates the magnitude and phase (in degrees) of the normalized input impedance. The magnitude plot utilizes the y-axis on the left, while the phase plot utilizes the y-axis on the right:

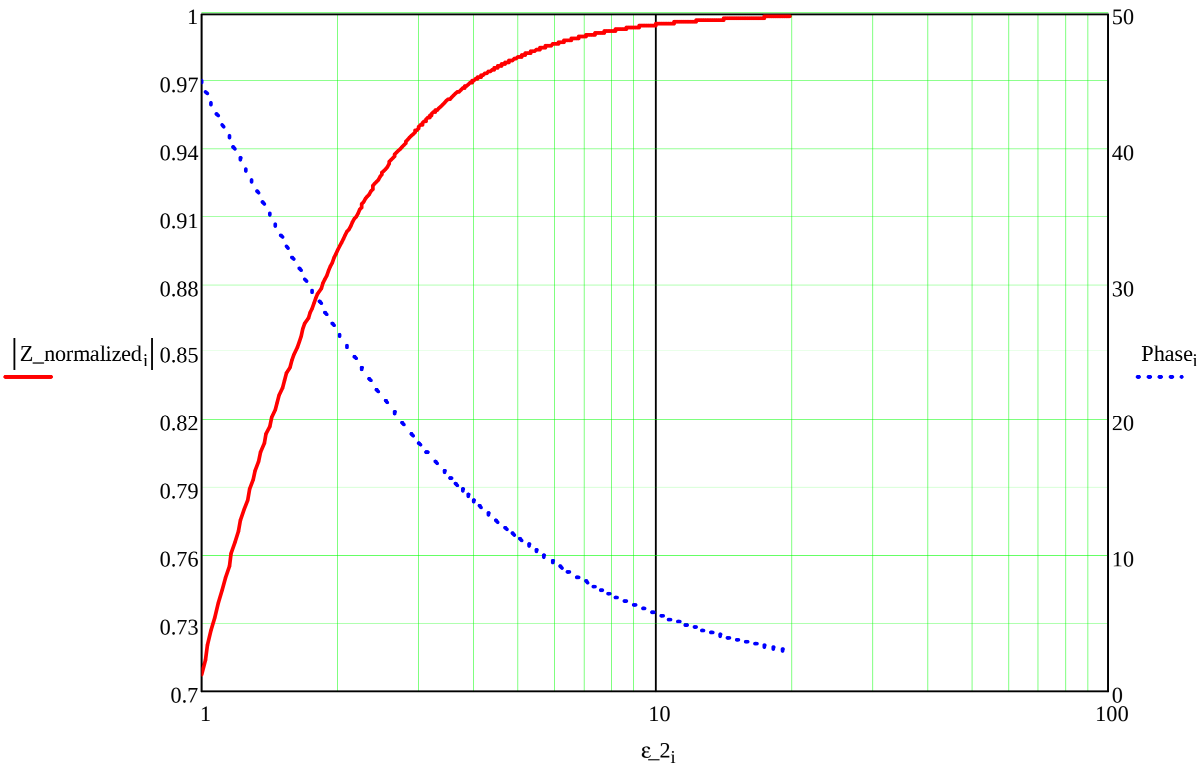

Figure 4 – Magnitude and Phase (degrees) of Normalized Input Impedance

The horizontal axis corresponds to ε_2 plotted over the range $1 \le \varepsilon_2 \le 20$. The magnitude plot indicated by the solid trace shows that for $\varepsilon_2 = 1$ the normalized impedance is equal to 0.7071, or $\sqrt{2}/2$, while the phase plot (indicated by the dotted trace) shows a phase angle of 45 degrees. Under these conditions, the transformer is operating in a way which departs significantly from its ideal behavior. The magnitude of the impedance is 30% below what is expected, and the transformer is introducing a 45 degree phase shift in the input impedance. As ε_2 increases the magnitude of the normalized input impedance rapidly approaches unity, while the phase angle steadily decreases. Observing the points on the graph that correspond to $\varepsilon_2 = 4$, it is seen that the magnitude of the normalized impedance is approximately equal to 0.97 while the phase angle is approximately 14 degrees. This data establishes a quantitative meaning of the “four times the impedance” guideline for broadband RF transformer windings:

When the inductive reactance of the secondary winding is at least four times the magnitude of the load impedance, the input impedance predicted by the “ideal” transformer equation will be within 3% of its actual value, with a phase error of no greater than 14 degrees.

Up to this point there has been little mention of any conditions placed on the primary winding. It is easy to extend this analysis to establish the same requirement for the primary winding by realizing that in an impedance matching application a requirement exists that the same impedance be seen looking in both directions at either set of transformer terminals. By looking backwards through the transformer towards the primary winding, it is clear that the load impedance wants to be matched to the impedance of the source driving it. Therefore, by looking at the smaller of the two impedances that need to be matched and designing the transformer so that the “four times the impedance” rule is satisfied at the lowest frequency of operation, one can be assured that the transformer will operate within 3% of its expected “ideal” performance.

Conclusion

In this article we have analyzed the impedance transformation properties of broadband RF transformers taking into account the finite inductive reactance of their primary and secondary windings. We have shown that the “four times the impedance” design guideline for the inductive reactance of the transformer windings yields an impedance transformation that departs from its “ideal” behavior by no more than 3%, and introduces a phase shift of no more than 14 degrees.

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